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Smearings of States Defined on Sharp Elements Onto Effect Algebras

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In some sense, a lattice effect algebra E is a smeared orthomodular lattice $S(E)$, which then becomes the set of all sharp elements of the effect algebra E. We show that if E is complete, atomic, and (o)-continuous, then a state on E exists iff there exists a state on $S(E)$. Further, it is shown that such an effect algebra E is an algebraic lattice compactly generated by finite elements of E. Moreover, every element of E has a unique basic decomposition into a sum of a sharp element and a $\oplus$-orthogonal set of unsharp multiples of atoms.

KEY WORDS: effect algebra; state; sharp elements; order-continuity.

1. INTRODUCTION

In "quantum probability theory" a carrier of a probability measure is a "quantum logic," which is an orthomodular lattice (or poset) if we assume noncompatible events; that means events that can be tested separately but not simultaneously (Kalmbach, 1983; Pták and Pulmanová, 1991). Recently, effect algebras have been introduced (Foulis and Bennett, 1994). The elements of an effect algebra represent quantum effects which are important for quantum measurements theory. In the fuzzy-probability setting the equivalent (in some sense) structure, D-poset was introduced by Kôpka (1992). Here elements represent fuzzy events which are statistical events that may not be crisp or sharp. Thus quantum events and fuzzy events have yes–no character that may be unsharp or imprecise (Gudder, 1998; Jenča and Riečanová, 1999; Riečanová, 2001a).

In general, an effect algebra is a partial algebra with two constants 0, 1 and a partial binary operation $\oplus$ (orthogonal sum) satisfying very simple axioms introduced in Section 2. Nevertheless, they are even finite effect algebras admitting no orthogonally additive measure and hence no states or probabilities (Greechie, 1971; Riečanová, 2001b). Some positive results are also known. For instance,

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on complete modular ortholattices and complete modular atomic effect algebras (see Kalmbach, 1983; Riečanová, 1998, 2001c). In every effect algebra $(E; \oplus, 0, 1)$ we can introduce a partial order by $a \leq b$ iff there is $c \in E$ with $a \oplus c = b$. If $(E; \leq)$ is a lattice, then E is called a lattice effect algebra, and if $(E; \leq)$ is a complete lattice, then E is called a complete effect algebra. In every lattice effect algebra the subset of all sharp elements (elements with $x \vee x' = 1$ or equivalently $x \wedge x' = 0$) is a sublattice being an orthomodular lattice (Jenča and Riečanová, 1999; Riečanová, 2001a). In this sense we may speak about lattice effect algebras as on smeared orthomodular lattices. It is important to note that every orthomodular lattice $(L; \vee, \wedge, ', 0, 1)$ can be itself organized into a lattice effect algebra if we define $a \oplus b = a \vee b$ iff $a \leq b'$. Then $(L; \oplus, 0, 1)$ is a lattice effect algebra in which $S(E) = E$. Also conversely. Hence a lattice effect algebra E is an orthomodular lattice iff $S(E) = E$.

Another important example of a lattice effect algebra can be derived from an MV-algebra $(M; \oplus, ', 0, 1)$ (Chang, 1958) if we define a partial binary operation $\widehat{\oplus}$ on M by $a \widehat{\oplus} b = a \oplus b$ iff $a \leq b'$. Then $(M; \widehat{\oplus}, 0, 1)$ is a lattice effect algebra in which $a \widehat{\oplus} (a' \wedge b) = b \widehat{\oplus} (b' \wedge a)$ for all $a, b \in E$ (then E is called an *MV*-effect algebra). Conversely, every lattice effect algebra $(E; \widehat{\oplus}, 0, 1)$ in which $a \widehat{\oplus} (a' \wedge b) = b \widehat{\oplus} (b' \wedge a)$ for all a, $b \in E$ can be organized into an *MV*-algebra by putting $a \oplus b = a \widehat{\oplus} (a' \wedge b)$ for all a, $b \in E$ (Bennett and Foulis, 1995; Kôpka and Chovanec, 1995; Lazar and Marinová, 2001). In every *MV*-effect algebra (*MV*-algebra) E we have $S(E) = C(E)$, where $C(E)$ is the center of E being a Boolean algebra. Thus in the above-mentioned sense an *MV*-effect algebra (*MV*-algebra) E is a smeared Boolean algebra. Moreover, E is a Boolean algebra iff $E = S(E)$.

The aim of this paper was to find an answer to the question: Does the existence of a state on the set $S(E)$ of sharp elements imply the existence of a state on the whole effect algebra E? Or, find some families of effect algebras having these properties.

We succeeded in finding a positive answer for all complete atomic (o)-continuous effect algebras and a negative answer for a finite (not lattice ordered) effect algebra. Actually, in Section 6 we introduced an example of a finite effect algebra admitting no states in spite of the fact that $S(E) = \{0, 1\}$. But the question stated above is at present far from being completely solved.

Finally, note that examples of lattice effect algebras which are neither orthomodular lattices nor *MV*-algebras are, for instance, a 0–1-pasting (horizontal sum) of two *MV*-algebras or a direct product of an orthomodular lattice and an *MV*-effect algebra. Here, instead of all these structures, we consider derived effect algebras.

2. BASIC DEFINITIONS AND FACTS

Definition 2.1. A structure $(E; \oplus, 0, 1)$ is called an *effect-algebra* if 0, 1 are two distinct elements and $\oplus$ is a partially defined binary operation on P, which satisfies the following conditions for any $a, b, c \in E$:

(i) $b \oplus a = a \oplus b$ if $a \oplus b$ is defined,
(ii) $(a \oplus b) \oplus c = a \oplus (b \oplus c)$ if one side is defined,
(iii) for every $a \in P$ there exists a unique $b \in P$ such that $a \oplus b = 1$ (we put $a' = b$), and
(iv) if $1 \oplus a$ is defined then $a = 0$.

We often denote the effect algebra $(E; \oplus, 0, 1)$ briefly by E. In every effect algebra E we can define the partial operation $\ominus$ and the partial order $\leq$ by putting

$$a \leq b \quad \text{and} \quad b \ominus a = c \qquad \text{iff } a \oplus c \text{ is defined and } a \oplus c = b.$$

Since $a \oplus c = a \oplus d$ implies $c = d$, the $\ominus$ and the $\leq$ are well defined. If E with the defined partial order is a lattice (a complete lattice), then $(E; \oplus, 0, 1)$ is called a *lattice effect algebra* (a *complete effect algebra*). For more details we refer the reader to Dvurečenskij and Pulmannová (2000) and the references given there. We review only a few properties without proof.

Lemma 2.2. *Elements of an effect algebra* $(E; \oplus, 0, 1)$ *satisfy the properties:*

(i) $a \oplus b$ *is defined iff* $a \leq b'$,
(ii) $a \leq a \oplus b$,
(iii) *if* $a \oplus b$ *and* $a \vee b$ *exist then* $a \wedge b$ *exist and* $a \oplus b = (a \wedge b) \oplus (a \vee b)$,
(iv) $a \oplus b \leq a \oplus c$ *iff* $b \leq c$ *and* $a \oplus c$ *is defined,*
(v) $a \ominus b = 0$ *iff* $a = b$, *and*
(vi) $a \leq b \leq c$ *implies that* $c \ominus b \leq c \ominus a$ *and* $b \ominus a = (c \ominus a) \ominus (c \ominus b)$.

If E *is a lattice effect algebra then*

(vii) $c \leq a, b \Longrightarrow (a \vee b) \ominus c = (a \ominus c) \vee (b \ominus c)$ *and* $(a \wedge b) \ominus c = (a \ominus c) \wedge (b \ominus c)$,
(viii) $a, b \leq c \Longrightarrow c \ominus (a \vee b) = (c \ominus a) \wedge (c \ominus b)$ *and* $c \ominus (a \wedge b) = (c \ominus a) \vee (c \ominus b)$,
(ix) $a, b \leq c' \Longrightarrow (a \oplus c) \vee (b \oplus c) = (a \vee b) \oplus c$ *and* $(a \wedge b) \oplus c = (a \oplus c) \wedge (b \oplus c)$.

It is worth noting that if $(E; \oplus, 0, 1)$ is an effect algebra, then $(E; \ominus, 0, 1)$ with the partial binary operation $\ominus$ defined above is a *D-poset* introduced by Kôpka and Chovanec (1994), and vice versa.

Recall that a set $Q \subseteq E$ is called a *subeffect algebra* of the effect algebra E if

(i) $1 \in Q$, and
(ii) if out of elements $a, b, c \in E$ with $a \ominus b = c$ two are in Q then $a, b, c \in Q$.

Assume that $(E_1; \oplus_1, 0_1, 1_1)$ and $(E_2; \oplus_2, 0_2, 1_2)$ are effect algebras. An injection $\varphi : E_1 \to E_2$ is called an *embedding* iff $\varphi(1_1) = 1_2$ and for $a, b \in E_1$ we have $a \leq b'$ iff $\varphi(a) \leq (\varphi(b))'$ in which case $\varphi(a \oplus_1 b) = \varphi(a) \oplus_2 \varphi(b)$. We can easily see that then $\varphi(E_1)$ is a subeffect algebra of E_2 and we say that E_1 and $\varphi(E_1)$ are *isomorphic*, or that E_1 *is up to isomorphism a subeffect algebra of* E_2. We usually identify E_1 with $\varphi(E_1)$.

We say that a finite system $F = (a_k)_{k=1}^n$ of not necessarily distinct elements of an effect algebra $(E; \oplus, 0, 1)$ is $\oplus$-orthogonal if $a_1 \oplus a_2 \oplus \cdots \oplus a_n$ (written $\bigoplus_{k=1}^n a_k$ or $\bigoplus F$) exists in E. Here we define $a_1 \oplus a_2 \oplus \cdots \oplus a_n = (a_1 \oplus a_2 \oplus \cdots \oplus a_{n-1}) \oplus a_n$ supposing that $\bigoplus_{k=1}^{n-1} a_k$ exists and $\bigoplus_{k=1}^{n-1} a_k \leq a_n'$. An arbitrary system $G = (a_\kappa)_{\kappa \in H}$ of not necessarily distinct elements of E is called $\oplus$-orthogonal if $\bigoplus K$ exists for every finite $K \subseteq G$. We say that for a $\oplus$-orthogonal system $G = (a_\kappa)_{\kappa \in H}$ the element $\bigoplus G$ exists iff $\bigvee\{\bigoplus K \mid K \subseteq G \text{ finite}\}$ exists in E and then we put $\bigoplus G = \bigvee\{\bigoplus K \mid K \subseteq G \text{ finite}\}$ (we write $G_1 \subseteq G$ iff there is $H_1 \subseteq H$ such that $G_1 = (a_\kappa)_{\kappa \in H_1}$).

An effect algebra $(E; \oplus, 0, 1)$ is called *Archimedean* if for no nonzero element $e \in E$ the elements

$$ne = \underbrace{e \oplus \ \oplus e \cdots \oplus e}_{n \text{ times}}$$

exist for all $n \in N$. An Archimedean effect algebra is called separable if every $\oplus$-*orthogonal* system of elements of E is at most countable. We can show that *every complete effect algebra is Archimedean* (Riečanová, 2000a).

For an element x of an effect algebra E we write $\text{ord}(x) = \infty$ if nx exists for every $n \in N$. We write $\text{ord}(x) = n_x \in N$ if n_x is the greatest integer such that $n_x x$ exists in E. Clearly, in an Archimedean effect algebra $n_x < \infty$ for every $x \in E$.

Elements x and y of a lattice effect algebra are called *compatible* (written $a \leftrightarrow b$) if $x \vee y = x \oplus (y \ominus (x \wedge y))$. If every two elements of E are compatible then E is called an *MV-effect algebra*. Every *MV*-effect algebra M can be organized into an *MV*-algebra by extending partial operation $\oplus$ onto the total binary operation $\widehat{\oplus}$ by putting $x \,\widehat{\oplus}\, y = x \oplus (x' \wedge y)$ for all $x, y \in M$. In a lattice effect algebra E every maximal subset $M \subseteq E$ of mutually compatible elements is a sublattice and a subeffect algebra of E. In fact M is an *MV*-effect algebra called block of E. Moreover, E is a union of its blocks (Riečanová, 2000b).

Lemma 2.3. *For elements of a lattice effect algebra E,*

(i) *if $x \wedge y = 0$ and for $m, n \in N$ the elements mx, ny, and $mx \oplus ny$ exist in E then $(kx) \wedge (ly) = 0$ and $(kx) \oplus (ly) = (kx) \vee (ly)$ for all $k \in \{1, \ldots, m\}$ and $l \in \{1, \ldots, n\}$,*

(ii) *if $Y \subseteq E$ with $\bigvee Y$ existing in E and $x \in E$ is such that $x \leftrightarrow y$ for all $y \in Y$ then $x \wedge (\bigvee Y) = \bigvee\{x \wedge y \mid y \in Y\}$ and $x \leftrightarrow \bigvee Y$.*

Proof: (i) If $(mx) \oplus (ny)$ is defined then $(kx) \oplus (ly)$ is defined for all $k \in \{1, \ldots, m\}, l \in \{1, \ldots, n\}$. By (i), $x \oplus y = (x \vee y) \oplus (x \wedge y) = x \vee y$ as $x \wedge y = 0$. By induction, supposing that $x \oplus (ly) = x \vee (ly)$ for all $l \in \{1, \ldots, n-1\}$, we obtain $x \oplus ((l+1)y) = (x \oplus (ly)) \oplus y = (x \vee (ly)) \oplus y = (x \oplus y) \vee ((l+1)y) = (x \vee y) \vee ((l+1)y) = x \vee (l+1)y$. It follows that $x \oplus (ly) = x \vee (ly)$ for all $l \in \{1, \ldots, n\}$. The last implies that $(ly) \oplus (kx) = (ly) \vee (kx)$ and hence $(kx) \wedge (ly) = 0$ for all $k \in \{1, \ldots, m\}$ and $l \in \{1, \ldots, n\}$.

(ii) For the proof we refer the reader to Jenča and Riečanová (1999). □

Definition 2.4. An element x of an effect algebra E is called *sharp* if $x \wedge x' = 0$. Set $S(E) = \{x \in E \mid x \wedge x' = 0\}$.

3. ATOMIC LATTICE EFFECT ALGEBRAS

A nonzero element of an effect algebra E is called an *atom* if $0 \le b < a$ implies $b = 0$. E is called *atomic* if for every nonzero element $x \in E$ there is an atom p of E such that $p \le x$. An element $u \in E$ is called finite if there is a finite system $(a_k)_{k=1}^n$ of not necessarily distinct atoms such that $u = \bigoplus_{k=1}^n a_\kappa$.

lemma 3.1 (Riečanová, 2001e)**.** Let $(E; \oplus, 0, 1)$ be an Archimedean atomic lattice effect algebra. Then for every nonzero $x \in E$

(i) *there is a* $\oplus$*-orthogonal system* $(a_\kappa)_{\kappa \in H}$ *of atoms of* E *such that* $x = \bigoplus_{\kappa \in H} a_\kappa$.
(ii) $x = \bigvee \{u \in E \mid u \le x, u$ *is finite*$\}$.

Recall that every lattice effect algebra E is *homogeneous*, i.e., for elements $a, b, c \in E$ with $a \le b \oplus c \le a'$ there are elements $u, v \in E$ such that $u \le b$, $v \le c$, and $a = u \oplus v$ (see Jenča, 2001). Evidently, if a, b, and c are atoms, then $a = b$ or $a = c$.

Lemma 3.1. *Assume that* $(E, \oplus, 0, 1)$ *is an atomic lattice effect algebra and* $a \in E$ *is an atom with* $\mathrm{ord}(a) = n_a \in N$. *Let* $S(E) = \{x \in E \mid x \wedge x' = 0\}$. *Then*

(i) $(ka) \wedge (ka)' \neq 0$ *for all* $\kappa \in \{1, 2, \ldots, n_a - 1\}$,
(ii) $n_a a \in S(E)$,
(iii) *if* $x \in E$ *with* $a \le x \le ka$ *then there is* $r \in N$ *such that* $x = ra$,
(iv) *If* $a, b \in E$ *are atoms and* $k, l \in N$ *are such that* $k \neq n_a$ *and* $ka = lb$ *then* $a = b$ *and* $k = l$,
(v) *if* $u = (k_1 a_1) \oplus (k_2 a_2) \oplus \cdots \oplus (k_n a_n)$ *where* $\{a_1, a_2, \ldots, a_n\}$ *is a set of mutually distinct atoms of* E *then* $u = \bigvee_{i=1}^n (k_i a_i)$.

For a proof we refer the reader to Riečanová (2001d). Note only that condition (iii) follows from the fact that elements $a, 2a, \ldots, n_a a$ and x are mutually compatible, and hence there is a block M of E containing these elements. Since a block is an MV-algebra, we obtain that $x = r\,a$ by the Riesz decomposition property (Cattaneo *et al.*, 2000). The proof of others is routine.

Theorem 3.3. *Let* $(E; \oplus, 0, 1)$ *be an Archimedean atomic lattice effect algebra. Then to every nonzero element* $x \in E$ *there are mutually distinct atoms* $a_\alpha \in \mathcal{E}$, $\alpha \in \mathcal{E}$, *and integers* k_α *such that*

$$x = \bigoplus\{k_\alpha a_\alpha \mid \alpha \in \mathcal{E}\} = \bigvee\{k_\alpha a_\alpha \mid \alpha \in \mathcal{E}\}$$

under which $x \in S(E)$ *iff* $k_\alpha = n_{a_\alpha} = \mathrm{ord}(a_\alpha)$ *for all* $\alpha \in \mathcal{E}$.

Proof: Assume $x \in E$, $x \neq 0$. By Riečanová (2001e) there is a $\oplus$-orthogonal system $G = (a_\varkappa)_{\varkappa \in H}$ of atoms of E such that $x = \bigoplus G$.

Let for every $\alpha \in H$, $K_\alpha = \{\varkappa \in H \mid a_\varkappa = a_\alpha\}$ and let $k_\alpha = |K_\alpha|$ be the cardinal number of K_α. As E is Archimedean, we have $k_\alpha \in N$ for every $\alpha \in H$. Moreover, for $\alpha, \beta \in H$ we have either $K_\alpha = K_\beta$ or $K_\alpha \cap K_\beta = \emptyset$. Let $\mathcal{E} = \{K_\alpha \mid \alpha \in H\}$. By axiom of choice there is a selection set $\{a_{\kappa_\alpha} \mid K_\alpha \in \mathcal{E}\}$ such that $a_{\kappa_\alpha} = a_\alpha$ for every $K_\alpha \in \mathcal{E}$. We put $G^* = \{k_\alpha a_{\kappa_\alpha} \mid K_\alpha \in \mathcal{E}\}$. Then to every finite system $F \subseteq G$ there is a finite set $F^* \subseteq G^*$ such that

$$\bigoplus F \leq \bigoplus \left\{k_\alpha a_{\kappa_\alpha} \mid k_\alpha a_{\kappa_\alpha} \in F^*\right\} = \bigvee \left\{k_\alpha a_{\kappa_\alpha} \mid k_\alpha a_{\kappa_\alpha} \in F^*\right\} \leq \bigoplus G.$$

Moreover, to every finite set $F^* \subseteq G^*$ there is a finite system $F \subseteq G$ such that $\bigoplus F^* = \bigoplus F$. As $x = \bigoplus G = \bigvee\{\bigoplus F \mid F \subseteq G, F \text{ is a finite system}\}$, we conclude that

$$x = \bigoplus \left\{k_\alpha a_{\kappa_\alpha} \mid K_\alpha \in \mathcal{E}\right\} = \bigvee \left\{k_\alpha a_{\kappa_\alpha} \mid K_\alpha \in \mathcal{E}\right\}.$$

Assume now that $x \in S(E)$. Then $x \wedge x' = 0$, which gives

$$0 = \left(\bigvee \left\{k_\alpha a_{\kappa_\alpha} \mid K_\alpha \in \mathcal{E}\right\}\right) \wedge \left(\bigwedge \left\{\left(k_\alpha a_{\kappa_\alpha}\right)' \mid K_\alpha \in \mathcal{E}\right\}\right).$$

Because there exists $(k_\alpha a_{\kappa_\alpha}) \oplus (k_\beta a_{\kappa_\beta})$ for all $K_\alpha \neq K_\beta$, we have $k_\alpha a_{\kappa_\alpha} \leq (k_\beta a_{\kappa_\beta})'$ and hence $k_\alpha a_{\kappa_\alpha} \leftrightarrow k_\beta a_{\kappa_\beta}$. Using Lemma 2.3 (ii), we obtain

$$0 = \bigvee_{K_\alpha \in \mathcal{E}} \left(k_\alpha a_{\kappa_\alpha} \wedge \bigwedge_{K_\beta \in \mathcal{E}} \left(k_\beta a_{\kappa_\beta}\right)'\right) = \bigvee_{K_\alpha \in \mathcal{E}} \left(k_\alpha a_{\kappa_\alpha}\right) \wedge \left(k_\alpha a_{\kappa_\alpha}\right)',$$

which gives that $(k_\alpha a_{\kappa_\alpha}) \wedge (k_\alpha a_{\kappa_\alpha})' = 0$ for all $K_\alpha \in \mathcal{E}$. It follows that $k_\alpha a_{\kappa_\alpha} \in S(E)$, which implies that $k_\alpha = \mathrm{ord}(a_{\kappa_\alpha})$ for every $K_\alpha \in \mathcal{E}$. □

4. COMPLETE ATOMIC (*o*)-CONTINUOUS EFFECT ALGEBRAS AND UNIQUE BASIC DECOMPOSITIONS OF ELEMENTS

The aim of this section is to show that every element in a complete atomic (*o*)-continuous effect algebra has a unique basic decomposition into a sum of a sharp element and a $\oplus$-orthogonal set of unsharp elements being multiples of atoms. It is also shown that every such effect algebra is an algebraic lattice compactly generated by finite elements.

We begin by recalling basic definitions.

Definition 4.1 (Grätzer, 1998). An element u of a complete lattice L is called *compact* if $u \leq \bigvee D$ for some $D \subseteq L$ implies that $u \leq \bigvee F$ for some finite $F \subseteq D$. A complete lattice L is called *algebraic* (or *compactly generated*) if every $x \in L$ is a join of compact elements of L.

Assume that $(\mathcal{E}; \prec)$ is a directed set and $(P; \leq)$ is a poset. A net of elements of P is denoted by $(a_\alpha)_{\alpha \in \mathcal{E}}$. If $a_\alpha \leq a_\beta$ for all $\alpha, \beta \in \mathcal{E}$ such that $\alpha \prec \beta$ then we write $a_\alpha \uparrow$. If moreover $a = \bigvee\{a_\alpha \mid \alpha \in \mathcal{E}\}$ we write $a_\alpha \uparrow a$. The meaning of $a_\alpha \downarrow$ and $a_\alpha \downarrow a$ is dual. For instance, $a \uparrow u_\alpha \leq v_\alpha \downarrow b$ means that $u_\alpha \leq v_\alpha$ for all $\alpha \in \mathcal{E}$ and $u_\alpha \uparrow a$ and $v_\alpha \downarrow b$. We will write $b \leq a_\alpha \uparrow a$ if $b \leq a_\alpha$ for all $\alpha \in \mathcal{E}$ and $a_\alpha \uparrow a$.

A net $(a_\alpha)_{\alpha \in \mathcal{E}}$ of elements of a poset $(P; \leq)$ *order converges* to a point $a \in P$ if there are nets $(u_\alpha)_{\alpha \in \mathcal{E}}$ and $(v_\alpha)_{\alpha \in \mathcal{E}}$ of elements of P such that

$$a \uparrow u_\alpha \leq a_\alpha \leq v_\alpha \downarrow a.$$

We write $a_\alpha \xrightarrow{(o)} a$ in P (or briefly $a_\alpha \xrightarrow{(o)} a$).

Definition 4.2. A lattice effect algebra $(E; \oplus, 0, 1)$ is called *order continuous* ((o)-*continuous* for brevity) if for any net of elements of E and $x, y \in E : x_\alpha \uparrow x \Rightarrow x_\alpha \wedge y \uparrow x \wedge y$.

It is easily seen that in an (o)-continuous lattice effect algebra $x_\alpha \xrightarrow{(o)} x$, $y_\alpha \xrightarrow{(o)} y \Rightarrow x_\alpha \vee y_\alpha \xrightarrow{(o)} x \vee y$ and $x_\alpha \wedge y_\alpha \xrightarrow{(o)} x \wedge y$. We need only consider that $x_\alpha \uparrow x$ iff $x'_\alpha \downarrow x'$ and that in every lattice $x_\alpha \uparrow x$, $y_\alpha \uparrow y \Rightarrow x_\alpha \vee y_\alpha \uparrow x \vee y$.

Theorem 4.3. *Every complete* (o)*-continuous atomic effect algebra is compactly generated by finite elements.*

Proof: Assume that $u = a_1 \oplus a_2 \oplus \cdots \oplus a_n$, where $a_1, \ldots, a_n$ are not necessarily different atoms of an effect algebra E. Further, let $G \subseteq E$ and $u \leq \bigvee G$. Let $\mathcal{E} = \{F \subseteq G \mid F \text{ is finite}\}$ be directed by set inclusion and let for every $F \in \mathcal{E}$ be $x_F = \bigvee F$. Then $x_F \uparrow x = \bigvee G$. As E is (o)-continuous, we have $a_1 \wedge x_F \uparrow x \wedge a_1 = a_1$, which implies that there is $F_1 \in \mathcal{E}$ such that for every $F \geq F_1$, $F \in \mathcal{E}$ we have $a_1 \leq x_F$. Further for $F \geq F_1$, $x_F \ominus a_1 \uparrow x \ominus a_1$ and hence $a_2 \wedge (x_F \ominus a_1) \uparrow a_2 \wedge (x \ominus a_1) = a_2$. It follows that there exists $F_2 \in \mathcal{E}$, $F_2 \geq F_1$ such that $a_2 \leq x_{F_2} \ominus a_1$, which gives that $a_1 \oplus a_2 \leq x_{F_2} \leq x_F$ for all $F \geq F_2$. By induction there are $F_\kappa \in \mathcal{E}, k = 1, 2, \ldots, n$ such that $F_n \geq F_{n-1} \geq \cdots \geq F_2 \geq F_1$ and $a_1 \oplus a_2 \oplus \cdots \oplus a_n \leq x_{Fn} = \bigvee F_n$. As F_n is a finite subset of G, this finishes the proof. □

Theorem 4.4. *The join of any two finite elements of a complete atomic* (o)*-continuous effect algebra* E *is a finite element.*

Proof: Assume that u and v are finite elements of E. By Theorem 3.3 there is a set of atoms $\{a_\alpha \mid \alpha \in \mathcal{E}\}$ and integers k_α such that $u \vee v = \bigoplus\{k_\alpha a_\alpha \mid \alpha \in \mathcal{E}\} = \bigvee\{k_\alpha a_\alpha \mid \alpha \in \mathcal{E}\}$. In view of Theorem 4.3 there are finite sets $F_1, F_2 \subseteq \mathcal{E}$ such that $u \leq \bigvee\{k_\alpha a_\alpha \mid \alpha \in F_1\}$ and $v \leq \bigvee\{k_\alpha a_\alpha \mid \alpha \in F_2\}$. It follows that $u \vee v \leq \bigvee\{k_\alpha a_\alpha \mid \alpha \in F_1 \cup F_2\} = \bigoplus\{k_\alpha a_\alpha \mid \alpha \in F_1 \cup F_2\} \leq \bigvee\{k_\alpha a_\alpha \mid \alpha \in \mathcal{E}\} = u \vee v$. We conclude that $u \vee v = \bigoplus\{k_\alpha a_\alpha \mid \alpha \in F_1 \cup F_2\}$, hence $u \vee v$ is a finite element of E. □

Theorem 4.5. *Let* E *be a complete atomic* (o)*-continuous effect algebra.*

(i) *If for two sets of atoms of* $E \{a_\alpha \mid \alpha \in \mathcal{A}\}$ *and* $\{b_\beta \mid \beta \in \mathcal{B}\}$ *and integers* $k_\alpha \neq \mathrm{ord}(a_\alpha)$ *and* $k_\beta \neq \mathrm{ord}(b_\beta)$ *it is satisfied*

$$\bigoplus\{k_\alpha a_\alpha \mid \alpha \in \mathcal{A}\} = \bigoplus\{l_\beta b_\beta \mid \beta \in \mathcal{B}\},$$

then for every $\alpha \in \mathcal{A}$ *there is* $\beta \in \mathcal{B}$ *such that* $a_\alpha = b_\beta$ *and* $k_\alpha = l_\beta$.

(ii) *For every* $x \in E$, $x \neq 0$ *there exists a unique* $w \in S(E)$, *a unique set* $\{a_\alpha \mid \alpha \in \mathcal{A}\}$ *of atoms of* E *and unique integers* $k_\alpha \neq \mathrm{ord}(a_\alpha)$ *such that*

$$x = \bigoplus\{k_\alpha a_\alpha \mid \alpha \in \mathcal{A}\} \oplus w.$$

Moreover, $(x \ominus w) \wedge w = 0$ *and if* $u \in S(E)$ *with* $u \leq x \ominus w$ *then* $u = 0$.

Proof: (i) Choose $\alpha_0 \in \mathcal{A}$. As $k_{\alpha_0} \neq \mathrm{ord}(a_{\alpha_0})$ we have $a_{\alpha_0} \leq k_{\alpha_0} a_{\alpha_0} \leq a'_{\alpha_0}$. Moreover, $k_\alpha a_\alpha \leq (k_{\alpha_0} a_{\alpha_0})' \leq a'_{\alpha_0}$ for every $\alpha \neq \alpha_0$, $\alpha \in \mathcal{A}$, because $(k_\alpha a_\alpha) \oplus (k_{\alpha_0} a_{\alpha_0})$ is defined. Thus, with the notation $x = \bigoplus\{k_\alpha a_\alpha \mid \alpha \in \mathcal{A}\}$ we have $a_{\alpha_0} \leq x \leq a'_{\alpha_0}$, which gives also $a_{\alpha_0} \leq \bigoplus\{l_\beta b_\beta \mid \beta \in \mathcal{B}\} = \bigvee\{l_\beta b_\beta \mid \beta \in \mathcal{B}\} \leq a'_0$. Since E is compactly generated by finite elements (Theorem 4.3), there is a finite set $F \subseteq \mathcal{B}$ such that

$$a_{\alpha_0} \leq \bigvee\{l_\beta b_\beta \mid \beta \in F\} = \bigoplus\{l_\beta b_\beta \mid \beta \in F\} \leq a'_{\alpha_0}.$$

It follows that there is $\beta_0 \in F$ such that $a_{\alpha_0} = b_{\beta_0}$, because E is homogeneous. Assume that $k_{\alpha_0} \neq l_{\beta_0}$. Without loss of generality we can assume that $k_{\alpha_0} < l_{\beta_0}$. Then

$$x \ominus (k_{\alpha_0} a_{\alpha_0}) = \bigoplus\{l_\beta b_\beta \mid \beta \neq \beta_0, \beta \in \mathcal{B}\} \oplus \left(l_{\beta_0} - k_{\alpha_0}\right) b_{\beta_0}.$$

As $b_{\beta_0} = a_{\alpha_0}$, we obtain that $a_{\alpha_0} \leq x \ominus (k_{\alpha_0} a_{\alpha_0}) \leq a'_{\alpha_0}$, which gives $a_{\alpha_0} \leq \bigoplus \{k_\alpha a_\alpha \mid \alpha \neq \alpha_0, \alpha \in \mathcal{A}\} \leq a'_{\alpha_0}$. In the same manner as above there is $\alpha_1 \in \mathcal{A}, \alpha_1 \neq \alpha_0$ such that $a_{\alpha_0} = a_{\alpha_1}$, a contradiction. This proves that $k_{\alpha_0} = l_{\beta_0}$. (ii) Set $w = \bigvee\{z \in S(E) \mid z \leq x\}$. Then $w \in S(E)$, as $S(E)$ is a complete lattice (Jenča and

Riečanová, 1999). Assume that $u \in S(E)$ and $u \leq x \ominus w$, which gives $u \leq x$. Then $u \leq w \wedge (x \ominus w) \leq w \wedge (1 \ominus w) = w \wedge w' = 0$, which gives $u = 0$.

Further, if $x \ominus w \neq 0$ then by Theorem 3.3 there is a set $\{a_\alpha \mid \alpha \in \mathcal{A}\}$ of atoms of E and there are integers k_α such that

$$x \ominus w = \bigoplus\{k_\alpha a_\alpha \mid \alpha \in \mathcal{A}\} = \bigvee\{k_\alpha a_\alpha \mid \alpha \in \mathcal{A}\}.$$

Assume that there is $\alpha \in \mathcal{A}$ such that $k_\alpha = \mathrm{ord}(a_\alpha)$. Then $k_\alpha a_\alpha \in S(E)$ and $k_\alpha a_\alpha \leq x \ominus w$, which gives $k_\alpha a_\alpha = 0$, a contradiction. Hence $k_\alpha \neq \mathrm{ord}(a_\alpha)$ for all $\alpha \in \mathcal{A}$ and in view of part (i) such set of atoms $\{a_\alpha \mid \alpha \in \mathcal{A}\}$ and integers $k_\alpha \neq \mathrm{ord}(a_\alpha)$ are unique. □

In the remainder of this paper we mean the equality in (ii) of Theorem 4.5 when we speak about a *unique basic decomposition of an element x of a complete atomic* (*o*)*-continuous effect algebra E*.

5. THE SMEARING THEOREM FOR STATES

Recall that a map $\omega : E \to [0, 1]$ is called a (finitely additive) *state* on an effect algebra $(E; \oplus, 0, 1)$ if $m(1) = 1$ and $x \leq y' \Rightarrow \omega(x \oplus y) = \omega(x) + \omega(y)$. A state is *faithful* if $\omega(x) = 0 \Rightarrow x = 0$. A state ω is called (*o*)*-continuous* (*order-continuous*) *if* $x_\alpha \xrightarrow{(o)} x \Rightarrow \omega(x_\alpha) \to \omega(x)$ for every net $(x_\alpha)_{\alpha \in \mathcal{E}}$ of elements of E.

Lemma 5.1. *A state* ω *on an effect algebra* E *is* (*o*)*-continuous iff* $x_\alpha \downarrow 0 \Rightarrow \omega(x_\alpha) \downarrow 0$ *for* $x_\alpha \in E$.

For a proof we refer the reader to Riečanová (2001c). Finally, recall that a map $\omega : L \to [0, 1]$ is a state on an orthomodular lattice $(L; \vee, \wedge, ', 0, 1)$ iff $\omega(\wedge) = \wedge$ and $\omega(x \vee y) = \omega(x) + \omega(y)$ for all $x \leq y'$, $x, y \in L$. Since for lattice effect algebra $(L; \oplus, 0, 1)$ derived from the orthomodular lattice L we have $x \oplus y = x \vee y$ iff $x \leq y'$, we conclude that ω is also a state on the effect algebra L.

For complete atomic (*o*)-continuous effect algebras, using Theorems 4.3 and 4.5, we can prove the following *Smearing Theorem* for states:

Theorem 5.2. *For every complete* (*o*)*-continuous atomic effect algebra* $(E; \oplus, 0, 1)$ *the following conditions are equivalent:*

(1) *There is a state on the orthomodular lattice* $S(E) = \{x \in E \mid x \wedge x' = 0\}$.
(2) *There is a state on* E.
(3) *There is an* (*o*)*-continuous state on* E.

Proof: (1) $\Rightarrow$ (2). Assume that a map $\omega : S(E) \to \langle 0, 1\rangle$ is a state on $S(E)$ and let us construct a state $\widehat{\omega}$ on E. To do this, we put for every atom $a \in E : \widehat{\omega}(a) = \frac{\omega(n_a a)}{n_a}$. As E is complete, which implies that E is Archimedean, we have $n_a < \infty$ hence $\widehat{\omega}(a)$ is well defined. Further, for a nonzero finite element $u \in E$ with the basic decomposition $u = (\bigoplus_{i=1}^{n} k_i a_i) \oplus w$ (Theorem 4.3) we put $\widehat{\omega}(u) = \sum_{i=1}^{n} k_i \widehat{\omega}(a_i) + \omega(w)$. Then for all finite $u, v \in E$ with $u \leq v'$ we have $\widehat{\omega}(u \oplus v) = \widehat{\omega}(u) + \widehat{\omega}(v)$, which is clear according to the unique basic decomposition of elements of E and the facts that for atoms $a, b, c \in E$ and $k \neq n_a$, $l \neq n_b$ we have $(ka) \oplus (lb) = n_c c$ iff $a = b = c$ and $k + l = n_c$, and that $S(E)$ is a subeffect algebra of E. This also gives that for finite elements of E we have $u_1 \leq u_2 \Rightarrow \widehat{\omega}(u_1) \leq \widehat{\omega}(u_2)$, because $u_2 \ominus u_1$ is also finite because of the fact that E is compactly generated by finite elements.

Let now $x \in E$, $x \neq 0$, and $\mathcal{U}_x = \{u, \in E \mid u \leq x, u$ is finite$\}$. Let $\mathcal{F} = \{F, \subseteq \mathcal{U}_x \mid F$ is a finite set$\}$ and let $u_F = \bigvee F$ for every $F \in \mathcal{F}$. By Theorem 4.4 every u_F is finite. Moreover, $\mathcal{F}$ is directed by set inclusion and $u_F \uparrow x$. We put $\widehat{\omega}(x) = \sup\{\widehat{\omega}(u_F) \mid F \in \mathcal{F}\}$.

Assume now that $x, y \in E$ are nonzero elements such that $x \leq y'$. Set $\mathcal{U}_x = \{u \in E \mid u \leq x, u$ is finite$\}$, $\mathcal{V}_y = \{v \in E \mid v \leq y, v$ is finite$\}$ and $\mathcal{F} = \{F \subseteq \mathcal{U}_x \cup \mathcal{V}_y \mid F$ is finite$\}$. Further let $u_F = \bigvee F \cap \mathcal{U}_x$ and $v_F = \bigvee F \cap \mathcal{V}_y$ for every $F \in \mathcal{F}$. Then $u_F \uparrow x$, $v_F \uparrow y$ and $u_F \oplus v_F \uparrow x \oplus y$ (see Riečanová, 2001e). Let us put $\mathcal{W}_{x \oplus y} = \{w \in E \mid w \leq x \oplus y, w$ is finite$\}$ and $w_D = \bigvee D$ for every finite $D \subseteq \mathcal{W}_{x \oplus y}$. Then $w_D \uparrow x \oplus y$ and because E is compactly generated by finite elements of E, for every w_D there is $F \in \mathcal{F}$ and finite $D^* \subseteq \mathcal{W}_{x \oplus y}$ such that $w_D \leq u_F \oplus v_F \leq w_{D^*}$, which gives $\widehat{\omega}(w_D) \leq \widehat{\omega}(u_F \oplus w_F) = \widehat{\omega}(u_F) + \widehat{\omega}(v_F) \leq \widehat{\omega}(w_{D^*})$. It follows that $\widehat{\omega}(x \oplus y) = \sup\{\widehat{\omega}(u_F) + \widehat{\omega}(v_F) \mid F \in \mathcal{F}\} = \widehat{\omega}(x) + \widehat{\omega}(y)$. This proves that $\widehat{\omega}$ is a state on E, because $0, 1 \in S(E)$, which implies that $\widehat{\omega}(0) = \omega(0) = 0$ and $\widehat{\omega}(1) = \omega(1) = 1$.

(1) $\Rightarrow$ (3). It suffices to show that the state $\widehat{\omega}$ constructed above is (o)-continuous. Assume that for a net $(x_\alpha)_{\alpha \in \mathcal{E}}$ of elements of E we have $x_\alpha \uparrow x$. As above, let $\mathcal{U}_x = \{u \in E \mid u \leq x, u$ is finite$\}$ and $u_F = \bigvee F$ for every finite set $F \subseteq \mathcal{U}_x$. Then $u_F \uparrow x$ and because E is compactly generated by finite elements of E, for every u_F there is a finite set $D_F \subseteq \mathcal{E}$ and $\alpha_F \in \mathcal{E}$ with $\alpha_F \geq \alpha$ for all $\alpha \in D_F$ and such that $u_F \leq \bigvee\{x_\alpha \mid \alpha \in D_F\} \leq x_{\alpha_F} \leq x$, and consequently $\widehat{\omega}(u_F) \leq \widehat{\omega}(x_{\alpha_F}) \leq \widehat{\omega}(x)$. Now using the definition of $\widehat{\omega}$, we obtain that $\widehat{\omega}(x) = \sup\{\widehat{\omega}(u_F) \mid F \subseteq \mathcal{U}_x, F$ is finite$\} = \sup\{\widehat{\omega}(x_\alpha) \mid \alpha \in \mathcal{E}\}$. This proves that $\widehat{\omega}$ is (o)-continuous.

(3)$\Rightarrow$ (2) $\Rightarrow$ (1). This is clear, because for $x, y \in S(E)$ with $x \leq y'$ we have $x \wedge y = 0$ and hence $x \vee y = x \oplus y$, by Lemma 2.3. Thus a restriction of a *state* ω defined on E onto $S(E)$ is a state on $S(E)$. □

Remark. Note that we have actually proved that if on the set $\mathcal{U}$ of all finite elements of a complete (o)-continuous atomic effect algebra E there is a bounded map $\omega : \mathcal{U} \to \langle 0, \infty)$ such that $\omega(0) = 0$ and $\omega(u \oplus v) = \omega(u) + \omega(v)$, for all $u \leq v'$, $u, v \in \mathcal{U}$, then there exists an (o)-continuous state on E.

Finally, it is clear that the state $\widehat{\omega}$ on E, constructed in the proof of Theorem 5.2, is an extension of a state ω defined on $S(E)$ iff ω is (o)-continuous. Further, if ω is faithful then $\widehat{\omega}$ is also faithful.

6. APPLICATIONS OF THE SMEARING THEOREM

In this section we indicate some families of effect algebras satisfying assumptions of the Smearing Theorem for states. We also introduce some applications.

6.1. Finite Lattice Effect Algebras

Theorem 5.2 can be applied on every finite lattice effect algebra E, because such E is evidently complete atomic and (o)-continuous. For instance, we obtain

On every finite lattice effect algebra E with $S(E)$ being a Boolean algebra there exists a (faithful) state.

On the other hand there is a finite effect algebra E (not lattice ordered) admitting no states in spite of the fact that $S(E) = \{0, 1\}$.

Example (Riečanová, 2001b). Let $E = \{0, a, b, c, 2a, 2b, 2c, 3b, 1\}$ be an effect algebra with $1 = a \oplus b \oplus c = 3a = 3c = 4b$ (Fig. 1). This equality implies that for a state ω on E should be $\omega(a) = \omega(c) = \frac{1}{3}$, $\omega(b) = \frac{1}{4}$, and $\omega(a) + \omega(b) + \omega(c) = 1$, a contradiction. Evidently here $S(E) = \{0, 1\}$.

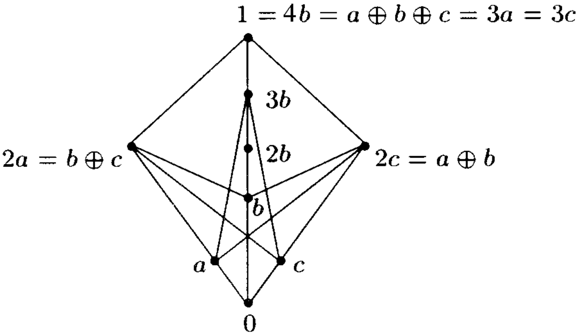

Fig. 1. An effect algebra admitting no states.

6.2. Profinite Effect Algebras

We call an effect algebra E profinite if it is a direct product of finite lattice effect algebras $E_\varkappa$. $\varkappa \in H$, where $H \neq \emptyset$ is an arbitrary. This means that E is a Cartesian product $\prod_{\varkappa \in H} E_\varkappa$ with "coordinatewise" defined $\oplus$, 0, and 1 and thus also $\leq$, $\vee$, and $\wedge$. Clearly, *every profinite effect algebra is complete, atomic, and* (o)-*continuous*, and hence it satisfies assumptions of Theorem 5.2. It is easy to check that a complete atomic effect algebra E is profinite iff the center $C(E)$ of E is atomic and there is only a finite set of elements of E under every atom of $C(E)$. More detailed. then E is isomorphic to the direct product $\prod_{\varkappa \in H}$. $[0, p_\varkappa]$, where $\{p_\varkappa \mid \varkappa \in H\}$ is the set of all atoms of $C(E)$ and $[0. p_\varkappa]$ for $\varkappa \in H$ is a finite lattice effect algebra with $\oplus$ inherited from E (see Riečanová, 2001d).

6.3. Complete Atomic (*o*)-Continuous Effect Algebras With *S*(*E*) Being a Boolean Algebra

In view of Lemma 3.2 and Theorem 3.3 we obtain that $S(E)$ is atomic. Thus if $S(E)$ is a Boolean algebra, then there exists a state on $S(E)$ which gives by Theorem 5.2 that

On every complete atomic (o)-*continuous effect algebra* E *with* $S(E)$ *being a Boolean algebra there exists an* (o)-*continuous state*.

Important example of such an effect algebra is every complete atomic *MV* -effect algebra (*MV* -algebra) E. It is because in an *MV* -effect algebra E we have $S(E) = C(E)$, where $C(E)$ is a center of E, which is a Boolean algebra (Greechie et al., 1995). Really, if $z \in C(E)$then $1 = (1 \wedge z) \vee (1 \wedge z') = z \vee z'$, which gives $z \wedge z' = 0$ and hence $z \in S(E)$. Conversely, if $z \in S(E)$ then $z \wedge z' = 0$, which gives $z \vee z' = 1$ and hence as E is distributive $x = 1 \wedge x = x \wedge (z \vee z') = (x \wedge z) \vee (x \wedge z')$, which $z \in C(E)$. Since every *MV*-effect algebra is (o)-continuous (by Lemma 2.3, (ii)) we obtain that

On every complete atomic MV-effect algebra (*MV-algebra*) *there is an* (o)-*continuous state*.

Further example of a lattice effect algebra with $S(E) = C(E)$ is every distributive effect algebra. By Theorem 5.2,

On every distributive complete atomic effect algebra there is an (o)-*continuous state*.

6.4. Complete Atomic Modular Effect Algebras

In Riečanová (2001d) it was shown that every complete atomic modular effect algebra E is (o)-continuous, hence E satisfies conditions of Theorem 5.2. Moreover, in such E the $S(E)$ is a complete modular atomic ortholattice. If E

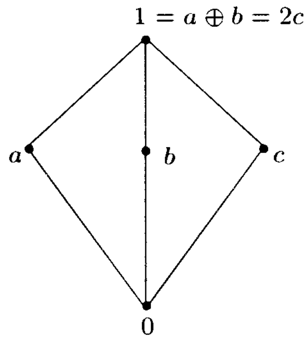

Fig. 2. Modular effect algebra admitting nonsubadditive states.

is separable then $S(E)$ is also separable and by Riečanová (1998) there is an (o)-continuous faithful state on $S(E)$. It follows, by Theorem 5.2 that there is an (o)-continuous faithful state on E (see Riečanová, 2001d).

7. CONCLUDING REMARKS

A state ω on a lattice effect algebra E is called a *valuation* if for all a, $b \in E\,\omega(a \vee b) + \omega(a \wedge b) = \omega(a) + \omega(b)$. It was proved in Riečanová (2001d) that a state ω on E is a valuation iff ω is subadditive, i.e., $\omega(a \vee b) \leq \omega(a) + \omega(b)$ for all $a, b \in E$. Moreover, if a faithful valuation on a lattice effect algebra E exists then E is modular. Next example shows that if in Theorem 5.2 a state ω on $S(E)$ is subadditive (hence a valuation) then the state $\widehat{\omega}$ constructed in the proof need not be subadditive on E even if E is modular.

Example. Let $E = \{0, a, b, c, 1\}$ be a modular effect algebra in which $1 = a \oplus b = 2c$ (Fig. 2). Then $S(E) = \{0, a, b, 1\}$ and so $\omega : S(E) \to [0, 1]$ such that $\omega(a) = \frac{1}{3}$, $\omega(b) = \frac{2}{3}$, $\omega(0) = 0$, and $\omega(1) = 1$ is a subadditive state on $S(E)$. But $\widehat{\omega} : E \to \langle 0, 1\rangle$ such that $\widehat{\omega}|S(E) = \omega$ and $\widehat{\omega}(c) = \frac{1}{2}$ is not subadditive because $\omega(a) + \omega(c) < \omega(a \vee c)$. Nevertheless, there is a unique valuation on E, namely $m(a) = m(b) = m(c) = \frac{1}{2}$ and $m(0) = 0$, $m(1) = 1$.

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